Outage Probability Bounds for Integer-Forcing Source Coding

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***Abstract*—Integer-forcing source coding has been proposed as a low complexity method for compression of distributed correlated Gaussian sources. In this scheme, each encoder quantizes its observation using the same fine lattice and reduces the result modulo the coarse lattice. Rather than directly recovering the individual quantized signals, the decoder first recovers a full-rank set of judiciously chosen integer linear combinations of the quantized signals, and then inverts it. It has been observed that the method works very well for "most" but not all source covariance matrices. The present work quantifies the measure of bad covariance matrices by studying the probability that integer forcing source coding fails as a function of the rate allocated in excess of the Berger-Tung benchmark, where the probability is with respect to a random orthogonal transformation that is applied to the sources prior to quantization. For the important case where the signals to be compressed correspond to the antenna inputs of relays in an i.i.d. Rayleigh fading environment, this orthogonal transformation can be viewed as if it is performed by nature. Hence, the results provide performance guarantees for distributed source coding via integer forcing in this scenario.**

I. INTRODUCTION

Integer-forcing source coding, proposed in [1], is a scheme for distributed lossy compression of correlated Gaussian sources under a minimum mean squared error distortion measure. Similar to its channel coding counterpart, in this scheme, all encoders use the same nested lattice codebook. Each encoder quantizes its observation using the fine lattice as a quantizer and reduces the result modulo the coarse lattice, which plays the role of binning. Rather than directly recovering the individual quantized signals, the decoder first recovers a full-rank set of judiciously chosen integer linear combinations of the quantized signals, and then inverts it.

An advantage of integer-forcing source coding over previously proposed practical methods (e.g., Wyner-Ziv) for the distributed source coding problem is its inherent symmetry, supporting equal distortion and quantization rates. This makes the scheme well suited for compressing observations of correlated relays or sensors [1]. As a concrete application example, consider a cloud radio access (C-RAN) network as depicted in Figure 1. Here, the base stations, which operate as radio units, are connected via fronthaul links to the managing control unit. The fronthaul links carry information about the baseband signals in the uplink from the radio units to the control unit. Due to the large bit rates produced by the quantized signals, compression prior to transmission on the fronthaul links is important. As the signals received by nodes may be highly correlated, and as the nodes may by physically separated, distributed compression is called for.

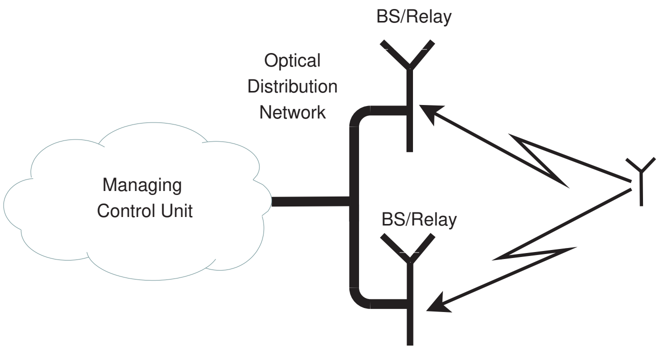

Fig. 1. Cloud radio access network communication scenario. Two basestations compress and forward the correlated signals they receive from several users.

Similar to IF channel coding, IF source coding works well for "most" but not all Gaussian vector sources. In the present work we quantify the measure of bad sources, following the approach of [2], by considering a randomized version of IF source coding where a random unitary transformation is applied to the sources prior to quantization. While in general such a transformation implies joint processing at the encoders, we note that in many natural scenarios, including that of C-RAN in an i.i.d. Rayleigh fading environment, the random transformation is actually performed by nature.[1] In fact, it was already empirically observed in [1] that IF source coding performs very well in the latter scenario.

II. PROBLEM FORMULATION AND BACKGROUND

In this section we recall the problem formulation and achievable rate of IF source coding as presented in [1].

A. Distributed Compression of Gaussian Sources

We start by outlining the classical problem of distributed lossy compression of jointly Gaussian random variables under a quadratic distortion measure. Specifically, we consider a distributed source coding setting with K encoding terminals and one decoder. Each of the K encoders has access to a vector $\mathbf{x}_k \in \mathbb{R}^n$ of n i.i.d. realizations of the random variable

[1]Specifically, note that the second order statistics of a random matrix with uncorrelated entries of equal variance remains unchanged by a unitary transformation applied at either side. As the distribution of a complex Gaussian matrix with i.i.d. entries is unaffected by such a transformation, it is readily seen that the (right and left) singular vectors of such a matrix follow the circular unitary distribution.

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x_k, $k = 1, ..., K$. The random vector $\mathbf{x} = [x_1 \ldots x_K]^T$ (corresponding to the different sources) is assumed to be Gaussian with zero mean and covariance matrix $\mathbf{K_{xx}} \triangleq \mathbb{E}\left(\mathbf{xx^T}\right)$.

Each encoder maps its observation $\mathbf{x_k}$ to an index using the encoding function

$$\mathcal{E}_k \ : \ \mathbb{R}^n \rightarrow \left\{1, ..., 2^{nR_k}\right\}, \quad (1)$$

and sends the index to the decoder.

The decoder is equipped with K decoding functions

$$\mathcal{D}_k \ : \ \left\{1, ..., 2^{nR_1}\right\} \times \cdots \times \left\{1, ..., 2^{nR_K}\right\} \rightarrow \mathbb{R}^n, \quad (2)$$

for $k = 1, ..., K$. Upon receiving K indices, one from each terminal, it generates the estimates

$$\hat{\mathbf{x}}_k = \mathcal{D}_k\left(\mathcal{E}_1(\mathbf{x}_1), ..., \mathcal{E}_K(\mathbf{x}_K)\right), \quad k = 1, ..., K. \quad (3)$$

A rate-distortion vector $(R_1, ..., R_K, d_1, ..., d_K)$ is achievable if there exist encoding functions $\mathcal{E}_1, ..., \mathcal{E}_K$ and decoding functions $\mathcal{D}_1, ..., \mathcal{D}_K$ such that $\frac{1}{n}\mathbb{E}\left(\|\mathbf{x}_k - \hat{\mathbf{x}}_k\|^2\right) \leq d_k$, for all $k = 1, ..., K$.

We focus on the symmetric case where $d_1 = \cdots = d_K = d$ and $R_1 = \cdots = R_K = R/K$, where we denote the sum rate by R. The best known achievable scheme (for this symmetric setting) is that of Berger and Tung [3], for which the following (in general, suboptimal) sum rate is achievable

$$\sum_{k=1}^{K} R_k \geq \frac{1}{2}\log\det\left(\mathbf{I} + \frac{1}{d}\mathbf{K_{xx}}\right) \triangleq R_{\mathrm{BT}}. \quad (4)$$

As shown in [1], R_{BT} is a lower bound on the achievable rate of IF source coding. We will refer to R_{BT} as the Berger-Tung benchmark. To simplify notation, we note that d can be "absorbed" into $\mathbf{K_{xx}}$. Hence, without loss of generality, we assume throughout that $d = 1$.

B. Compound Source Model And Scheme Outage Formulation

We define the following compound class of Gaussian sources that have the same value of R_{BT} via their covariance matrix:

$$\mathbb{K}(R_{\mathrm{BT}}) = \left\{\mathbf{K_{xx}} \in \mathbb{R}^{K\times K} : \log\det\left(\mathbf{I} + \mathbf{K_{xx}}\right) = R_{\mathrm{BT}}\right\}. \quad (5)$$

We quantify the measure of the set of bad sources by considering outage events, i.e., those events (sources) where integer forcing fails to achieve the desired level of distortion even though the rate exceeds R_{BT}. More broadly, for a given quantization scheme, denote the necessary rate to achieve $d = 1$ for a given covariance matrix $\mathbf{K_{xx}}$ as $R_{\mathrm{scheme}}(\mathbf{K_{xx}})$. Then, given a target rate $R > R_{BT}$ and a covariance matrix $\mathbf{K_{xx}} \in \mathbb{K}(R_{\mathrm{BT}})$, a scheme outage occurs when $R_{\mathrm{scheme}}(\mathbf{K_{xx}}) > R$.

To quantify the measure of this event, we follow [2] and apply random orthogonal precoding to the samples of sources prior to encoding. As mentioned above, this amounts to joint processing of the samples and hence the problem is no longer distributed in general. Nonetheless, we describe in the sequel scenarios where this precoding is redundant as it can be viewed as being performed by nature in a statistical setting.

Applying precoding to the samples of the source, we obtain a transformed source vector $\tilde{\mathbf{x}} = \mathbf{Px}$, with covariance matrix $\mathbf{K}_{\tilde{x}\tilde{x}} = \mathbf{PK}_{xx}\mathbf{P}^T$.

It follows that the achievable rate of a quantization scheme for the precoded source is $R_{\mathrm{scheme}}(\mathbf{K}_{\tilde{x}\tilde{x}})$. When $\mathbf{P}$ is drawn at random, the latter rate is also random. The worst-case (WC) scheme outage probability is defined in turn as

$$P_{\mathrm{out,scheme}}^{\mathrm{WC}}\left(R_{\mathrm{BT}}, \Delta R\right) = \sup_{\mathbf{K_{xx}} \in \mathbb{K}(R_{\mathrm{BT}})} \Pr\left(R_{\mathrm{scheme}}(\mathbf{K}_{\tilde{x}\tilde{x}}) > R_{\mathrm{BT}} + \Delta R\right), \quad (6)$$

where the probability is over the ensemble of orthogonal precoding matrices considered and ΔR is the gap to the Berger-Tung benchmark. The goal of this paper is to quantify the tradeoff between the quantization rate R (or equivalently its distance ΔR from R_{BT}) and the outage probability $P_{\mathrm{out,IF}}^{\mathrm{WC}}\left(R_{\mathrm{BT}}, \Delta R\right)$.

C. Integer-forcing Source Coding

In a manner similar to IF equalization, IF can be applied to the problem of distributed lossy compression. The approach is based on standard quantization followed by lattice-based binning. However, in the IF framework, the decoder first uses the bin indices for recovering linear combinations with integer coefficients of the quantized signals, and only then recovers the quantized signals themselves.

For our purposes it suffices to state only the achievable rates of IF source coding. We refer the reader to [1] for the derivation and proofs. We recall Theorem 1 from [1], which states that for any covariance matrix $\mathbf{K_{xx}}$, IF source coding can achieve any (sum) rate satisfying

$$R > R_{\mathrm{IF}} \triangleq \frac{K}{2}\log\left(\min_{\substack{\mathbf{A}\in\mathbb{Z}^{K\times K} \\ \det\mathbf{A}\neq 0}} \max_{k=1,\ldots,K} \mathbf{a}_k^T\left(\mathbf{I} + \mathbf{K_{xx}}\right)\mathbf{a}_k\right), \quad (7)$$

where $\mathbf{a}_k^T$ is the kth row of the matrix $\mathbf{A}$.

The matrix $\mathbf{I} + \mathbf{K_{xx}}$ is symmetric and positive definite, and therefore it admits a Cholesky decomposition

$$\mathbf{I} + \mathbf{K_{xx}} = \mathbf{FF}^T. \quad (8)$$

With this notation, we have

$$R_{\mathrm{IF}} = \frac{K}{2}\log\left(\min_{\substack{\mathbf{A}\in\mathbb{Z}^{K\times K} \\ \det\mathbf{A}\neq 0}} \max_{k=1,\ldots,K} \|\mathbf{F}^T\mathbf{a_k}\|^{\mathbf{2}}\right). \quad (9)$$

Denote by $\Lambda(\mathbf{F}^T)$ the K-dimensional lattice spanned by the matrix $\mathbf{F}^T$, i.e., $\Lambda(\mathbf{F}^T) \triangleq \left\{\mathbf{F}^T\mathbf{a} : \ \mathbf{a} \in \mathbb{Z}^{\mathbf{K}}\right\}$.

It follows that the problem of finding the optimal matrix $\mathbf{A}$ is equivalent to finding the K linearly independent shortest vectors of $\Lambda(\mathbf{F}^T)$. Denoting the k-successive minimum of the lattice by $\lambda_k^2(\mathbf{F}^T)$, we have

$$R_{\mathrm{IF}} = \frac{K}{2}\log\left(\lambda_K^2(\mathbf{F}^T)\right). \quad (10)$$

Just as successive interference cancellation significantly improves the achievable rate of integer-forcing equalizer in

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channel coding, an analogous scheme can be implemented in the case of IF source coding. Specifically, let $\mathbf{L}$ be defined by the Cholesky decomposition

$$\mathbf{A}\left(\mathbf{I}+\mathbf{K}_{\mathbf{xx}}\right)\mathbf{A}^T=\mathbf{L}\mathbf{L}^T. \tag{11}$$

Then, as shown in [4] and [5], the achievable rate of IF source coding with successive noise prediction (which we denote as IF-SUC) is given by

$$R_{\text{IF}-\text{SUC}}=\frac{K}{2}\max_{m=1,\ldots,K}\log\left(l_{m,m}^2\right). \tag{12}$$

While the gap between R_{IF} (and even more so $R_{\text{IF}-\text{SUC}}$) and R_{BT} is quite small for most covariance matrices, it can nevertheless be arbitrarily large. We next quantify the measure of bad covariance matrices by considering randomly precoded IF source coding.

III. Upper Bound on the Scheme Outage for Random Unitary Precoding

Recalling (7), we have that for the IF scheme with a given precoding matrix $\mathbf{P}$

$$R_{\text{P}-\text{IF}}(\mathbf{K}_{xx},\mathbf{P})=\frac{K}{2}\log\left(\min_{\substack{\mathbf{A}\in\mathbb{Z}^{K\times K}\\ \det\mathbf{A}\neq 0}}\max_{k=1,\ldots,K}\mathbf{a}_k^T\left(\mathbf{I}+\mathbf{P}\mathbf{K}_{xx}\mathbf{P}^T\right)\mathbf{a}_k\right). \tag{13}$$

Since $\mathbf{K}_{\mathbf{xx}}$ is symmetric, it allows orthonormal diagonalization

$$\left(\mathbf{I}+\mathbf{K}_{\mathbf{xx}}\right)=\mathbf{U}\mathbf{D}\mathbf{U}^T. \tag{14}$$

When orthogonal precoding is applied, we have

$$\left(\mathbf{I}+\mathbf{P}\mathbf{K}_{\mathbf{xx}}\mathbf{P}^T\right)=\mathbf{P}\mathbf{U}\mathbf{D}\mathbf{U}^T\mathbf{P}^T. \tag{15}$$

To bound the measure of "bad" sources, we use precoded matrices drawn from the circular real ensemble (CRE). The ensemble is defined by the unique distribution on orthogonal matrices that is invariant under left and right orthogonal transformations [6]. That is, given a random matrix $\mathbf{P}$ drawn from the CRE, for any orthogonal matrix $\grave{\mathbf{U}}$, both $\mathbf{P}\grave{\mathbf{U}}$ and $\grave{\mathbf{U}}\mathbf{P}$ are equal in distribution to $\mathbf{P}$.

Since $\mathbf{P}\mathbf{U}^T$ is equal in distribution to $\mathbf{P}$ with CRE precoding, for the sake of computing outage probabilities, we may simply assume that $\mathbf{U}^T$ (and also $\mathbf{U}$) is drawn from the CRE. It follows that for a specific choice of $\mathbf{a_k}$, we have

$$\begin{aligned}R_{P-IF}(\mathbf{D},\mathbf{U},\mathbf{a}_k)&=\frac{1}{2}\log\left(\mathbf{a}_k^T\mathbf{U}\mathbf{D}\mathbf{U}^T\mathbf{a}_k\right)\\&=\frac{1}{2}\log\left(\|\mathbf{D}^{1/2}\mathbf{U}^T\mathbf{a}_k\|^2\right).\end{aligned} \tag{16}$$

Hence,

$$R_{\text{P}-\text{IF}}(\mathbf{D},\mathbf{U})=\frac{K}{2}\log\left(\min_{\substack{\mathbf{A}\in\mathbb{Z}^{K\times K}\\ \det\mathbf{A}\neq 0}}\max_{k=1,\ldots,K}\log\left(\|\mathbf{D}^{1/2}\mathbf{U}^T\mathbf{a}_k\|^2\right)\right). \tag{17}$$

Let Λ be the lattice spanned by $\mathbf{G}=\mathbf{D}^{1/2}\mathbf{U}^T$. Then (17) may be rewritten as

$$R_{\text{P}-\text{IF}}(\mathbf{D},\mathbf{U})=\frac{K}{2}\log\left(\lambda_K^2(\Lambda)\right). \tag{18}$$

Let $\mathbb{D}(R_{\text{BT}})$ denote the set of all diagonal matrices $\mathbf{D}$, such that $\det(\mathbf{D})=2^{2R_{\text{BT}}}$. We may thus rewrite the worst-case IF outage probability as defined in (6) as

$$P_{\text{out,IF}}^{\text{WC}}\left(R_{\text{BT}},\Delta R\right)=\sup_{\mathbf{D}\in\mathbb{D}(R_{\text{BT}})}\Pr\left(R_{\text{IF}}(\mathbf{D},\mathbf{U})>R_{\text{BT}}+\Delta R\right), \tag{19}$$

where probability is calculated with respect to matrix $\mathbf{U}$ which is drawn from the CRE.

The next lemma provides an upper bound on the outage probability as a function of the rate gap ΔR, R_{BT}, and $d_{\max}$ (as well as the number of sources). Denote

$$\mathbb{A}(\beta,\delta)=\left\{\mathbf{a}:0<\|\mathbf{a}\|<\sqrt{\frac{\beta}{\delta}}\right\}. \tag{20}$$

Lemma 1: For any K sources such that $\mathbf{D}\in\mathbb{D}(R_{\text{BT}})$, and for $\mathbf{U}$ drawn from the CRE, we have

$$\Pr\left(R_{\text{P}-\text{IF}}(\mathbf{D},\mathbf{U})<R_{\text{BT}}+\Delta R\right)<\sum_{\mathbf{a}\in\mathbb{A}(\beta,1/d_{\max})}\frac{K\left(\frac{K+3}{4}{\gamma_K^*}^2\right)^{\frac{K-1}{2}}2^{-\frac{K-1}{K}(R_{\text{BT}}+\Delta R)}}{\frac{\|\mathbf{a}\|^{K-1}}{2^{R_{BT}}}\sqrt{d_{\max}}}. \tag{21}$$

where $d_{\max}=\max_i\mathbf{D}_{i,i}$, γ_i is Hermite constant, $\gamma_K^*=\max_{i=1,\ldots,K}\gamma_i$ and

$$\mathbb{A}(\beta,1/d_{\max})=\left\{\mathbf{a}:0<||\mathbf{a}||<\sqrt{\beta d_{\max}}\right\}. \tag{22}$$

Proof: Let Λ denote the dual lattice of Λ^* and note that it is spanned by the matrix

$$(\mathbf{G}^T)^{-1}=\mathbf{D}^{-1/2}\mathbf{U}^T. \tag{23}$$

The successive minima of Λ and Λ^* are related by (Theorem 2.4 in [7])

$$\lambda_1(\Lambda^*)^2\lambda_K(\Lambda)^2\leq\frac{K+3}{4}{\gamma_K^*}^2, \tag{24}$$

where γ_K is Hermite's constant. Therefore, we may express the achievable rates of P-IF via the dual lattice as follows

$$R_{\text{P}-\text{IF}}(\mathbf{D},\mathbf{U})\leq\frac{K}{2}\log\left(\frac{K+3}{4}{\gamma_K^*}^2\frac{1}{\lambda_1(\Lambda^*)^2}\right). \tag{25}$$

Hence, we have

$$\begin{aligned}&\Pr\left(R_{\text{P}-\text{IF}}(\mathbf{D},\mathbf{U})>R_{\text{BT}}+\Delta R\right)\leq\\&\Pr\left(\frac{K}{2}\log\left(\frac{K+3}{4}{\gamma_K^*}^2\frac{1}{\lambda_1(\Lambda^*)^2}\right)>R_{\text{BT}}+\Delta R\right)=\\&\Pr\left(\lambda_1(\Lambda^*)^2<\frac{K+3}{4}{\gamma_K^*}^2 2^{-\frac{2}{K}(R_{\text{BT}}+\Delta R)}\right)\end{aligned} \tag{26}$$

Let $\beta=\frac{K+3}{4}{\gamma_K^*}^2 2^{-\frac{2}{K}(R_{\text{BT}}+\Delta R)}$. We wish to bound (26), or equivalently, we wish to bound

$$\Pr\left(\lambda_1^2(\Lambda^*) < \beta\right) = \Pr\left(\lambda_1(\Lambda^*) < \sqrt{\beta}\right) \tag{27}$$

for a given matrix $\mathbf{D}$. Note that the event $\lambda_1(\Lambda^*) < \sqrt{\beta}$ is equivalent to the event

$$\bigcup_{\mathbf{a}\in\mathbb{Z}^{2N_t}\setminus\{\mathbf{0}\}} ||\mathbf{D}^{-1/2}\mathbf{U}^T\mathbf{a}|| < \sqrt{\beta}. \tag{28}$$

Applying the union bound yields

$$\Pr\left(\lambda_1(\Lambda^*) < \sqrt{\beta}\right) < \sum_{\mathbf{a}\in\mathbb{Z}^{2N_t}} \Pr\left(||\mathbf{D}^{-1/2}\mathbf{U}^T\mathbf{a}|| < \sqrt{\beta}\right). \tag{29}$$

Note that if $\frac{||\mathbf{a}||}{\sqrt{d_{\max}}} \geq \sqrt{\beta}$, we have

$$\Pr\left(||\mathbf{D}^{-1/2}\mathbf{U}^T\mathbf{a}|| < \sqrt{\beta}\right) = 0. \tag{30}$$

Therefore, using (20), the set of relevant vectors $\mathbf{a}$ is

$$\mathbb{A}(\beta, 1/d_{\max}) = \left\{\mathbf{a} : 0 < ||\mathbf{a}|| < \sqrt{\beta d_{\max}}\right\}. \tag{31}$$

It follows from (29) and (30) that

$$\Pr\left(\lambda_1(\Lambda^*) < \sqrt{\beta}\right) < \sum_{\mathbf{a}\in\mathbb{A}(\beta,1/d_{\max})} \Pr\left(\|\mathbf{D}^{-1/2}\mathbf{U}^T\mathbf{a}\| < \sqrt{\beta}\right). \tag{32}$$

Following the footsteps of Lemma 2 in [2] and adopting the same geometric interpretation described there, we may interpret $\Pr\left(\|\mathbf{D}^{-1/2}\mathbf{U}^T\mathbf{a}\| < \sqrt{\beta}\right)$ as the ratio of the surface area of an ellipsoid that is inside a ball with radius $\sqrt{\beta}$ and the surface area of the entire ellipsoid. The axes of this ellipsoid are defined as $x_i = \frac{\|\mathbf{a}\|}{\sqrt{d_i}}$.

Denote the vector $\mathbf{o}_{\|\mathbf{a}\|}$ as a vector drawn from the CRE with norm $\|\mathbf{a}\|$. Using Lemma 1 in [2] and since we assume that $\mathbf{U}^T$ is drawn from the CRE, we have that the right hand side of (32) is equal to

$$\sum_{\mathbf{a}\in\mathbb{A}(\beta,1/d_{\max})} \Pr\left(\|\mathbf{D}^{-1/2}\mathbf{o}_{\|\mathbf{a}\|} < \sqrt{\beta}\right) = \sum_{\mathbf{a}\in\mathbb{A}(\beta,1/d_{\max})} \frac{\mathrm{CAP}_{\mathrm{ell}}(\mathrm{x}_1,\cdots,\mathrm{x}_\mathrm{K})}{L(x_1,\cdots,x_K)} \tag{33}$$

where

$$\mathrm{CAP}_{\mathrm{ell}}(\mathrm{x}_1,\cdots,\mathrm{x}_\mathrm{K}) < K\frac{\pi^{K/2}}{\Gamma(1+K/2)}\sqrt{\beta}^{K-1} \triangleq \overline{\mathrm{CAP}_{\mathrm{ell}}}, \tag{34}$$

and

$$L(x_1,\cdots,x_K) > \frac{\pi^{K/2}}{\Gamma(1+K/2)}\frac{\|\mathbf{a}\|^K}{\prod_{i=1}^K\sqrt{d_i}}\sum_{i=1}^K\frac{\sqrt{d_i}}{\|\mathbf{a}\|} \tag{35}$$

$$> \frac{\pi^{K/2}}{\Gamma(1+K/2)}\frac{\|\mathbf{a}\|^{K-1}\sqrt{d_{\max}}}{2^{R_{BT}}} \triangleq \underline{L}. \tag{36}$$

Substituting (34) and (36) in (33), we obtain

$$\sum_{\mathbf{a}\in\mathbb{A}(\beta,1/d_{\max})} \frac{\mathrm{CAP}_{\mathrm{ell}}(\mathrm{x}_1,\cdots,\mathrm{x}_\mathrm{K})}{L(x_1,\cdots,x_K)} < \sum_{\mathbf{a}\in\mathbb{A}(\beta,1/d_{\max})} \frac{\overline{\mathrm{CAP}_{\mathrm{ell}}}}{\underline{L}}$$

$$= \sum_{\mathbf{a}\in\mathbb{A}(\beta,1/d_{\max})} \frac{K\sqrt{\beta}^{K-1}}{\frac{\|\mathbf{a}\|^{K-1}}{2^{R_{BT}}}\sqrt{d_{\max}}}. \tag{37}$$

Recalling that $\beta = \frac{K+3}{4}{\gamma_K^*}^2 2^{-\frac{2}{K}(R_{\mathrm{BT}}+\Delta R)}$, we get that

$$\Pr\left(R_{\mathrm{P-IF-SUC}}(\mathbf{D},\mathbf{U}) > C + \Delta R\right) < \sum_{\mathbf{a}\in\mathbb{A}(\beta,1/d_{\max})} \frac{K\left(\frac{K+3}{4}{\gamma_K^*}^2\right)^{\frac{K-1}{2}} 2^{-\frac{K-1}{K}(R_{\mathrm{BT}}+\Delta R)}}{\frac{\|\mathbf{a}\|^{K-1}}{2^{R_{BT}}}\sqrt{d_{\max}}}. \tag{38}$$

■

While Lemma 1 provides an explicit bound on the outage probability, in order to calculate it, one needs to go over all diagonal matrices in $\mathbb{D}(R_{\mathrm{BT}})$ and for each such diagonal matrix, sum over all the relevant integer vectors in $\mathbb{A}(\beta, d_{\min})$. Hence, the bound can be evaluated only for moderate compression rates and for a small number of sources. The following theorem provides a looser, yet very simple closed-form bound. Another advantage for this bound is that it does not depend on the rate.

Theorem 1: For any K sources such that $\det(\mathbf{D}) = 2^{2R_{\mathrm{BT}}}$, and for $\mathbf{U}$ drawn from the CRE we have

$$\Pr\left(R_{\mathrm{-IF-SUC}}(\mathbf{D},\mathbf{U}) > C + \Delta R\right) < c(K)2^{-\Delta R}, \tag{39}$$

where

$$c(K) = K\left(\frac{K+3}{4}{\gamma_K^*}^2\right)^{\frac{K}{2}}\left(1+\sqrt{K}\right)^K\frac{\pi^{K/2}}{\Gamma(K/2+1)}. \tag{40}$$

Note that $c(K)$ is a constant that depends only on the number of sources K.

Proof: See Appendix A in [8]. ■

Similarly to the case of channel coding, analyzing Theorem 1 shows there are two main sources for looseness that may be further tightened:

- *Union bound* - While there is an inherent loss in the union bound, in fact, some terms in the summation (32) may be completely dropped.[2] Specifically, using Corollary 1 in [2], the set $\mathbb{A}(\beta, 1/d_{\max})$ appearing in the summation in (1) may be replaced by the smaller set $\mathbb{B}(\beta, 1/d_{\max})$ where

$$\mathbb{B}(\beta,d) = \left\{\mathbf{a} : 0 < \|\mathbf{a}\| < \sqrt{\frac{\beta}{d}} \text{ and } \nexists 0 < c < 1 \text{ s.t. } c\mathbf{a}\in\mathbb{Z}^n\right\}. \tag{41}$$

- *Dual Lattice* - Bounding via the dual lattice induces a loss reflected in (24). This may be circumvented for the case of two sources, as accomplished in Lemma 2 and Theorem 2 below by using IF-SUC.

Lemma 2: For any two sources with Berger-Tung benchmark R_{BT}, i.e., $\mathbf{D}\in\mathbb{D}(R_{\mathrm{BT}})$, and for $\mathbf{U}$ drawn from the CRE we have

$$\Pr\left(R_{\mathrm{P-IF-SUC}}(\mathbf{D},\mathbf{U}) > R_{\mathrm{BT}} + \Delta R\right) <$$

[2]Similar to [2], a simple factor of 2 can be deduced (regardless of the mutual information and number of sources) by noting that $\mathbf{a}$ and $-\mathbf{a}$ result in the same outcome and hence there is no need to check both cases.

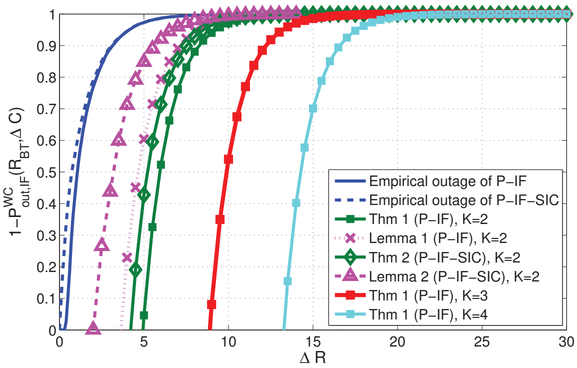

Fig. 2. Outage bounds for different number of sources.

$$\sum_{\mathbf{a}\in\mathbb{A}(\beta,d_{\min})} \frac{2\sqrt{\beta}}{\|\mathbf{a}\|2^{R_{\mathrm{BT}}}\frac{1}{\sqrt{d_{\min}}}}. \quad (42)$$

where $d_{\min} = \min_i \mathbf{D}_{i,i}$ and

$$\mathbb{A}(\beta, d_{\min}) = \left\{ \mathbf{a} : 0 < \|\mathbf{a}\| < \sqrt{\frac{\beta}{d_{\min}}} \right\}. \quad (43)$$

Theorem 2: For any two sources such that $\det(\mathbf{D}) = 2^{2R_{\mathrm{BT}}}$, and for $\mathbf{U}$ drawn from the CRE, we have

$$\Pr\left(R_{\mathrm{P-IF-SUC}}(\mathbf{D},\mathbf{U}) > C + \Delta R\right) < c(K)2^{-\Delta R}, \quad (44)$$

where

$$c(K) = 2\left(1+\sqrt{2}\right)^2 \pi. \quad (45)$$

Proof: See Appendix B in [8]. ∎

Figure 2 depicts the bounds derived as well as results of a Monte Carlo evaluation of (6) for the case of 2 sources and CRE precoding.When calculating the empirical curves and the lemmas, we assumed high quantization rates ($R_{\mathrm{BT}} = 14$). The lemmas were calculated by going over a grid of values of d_1 and d_2.

IV. Application: Distributed Compression for C-RAN

Since we described IF source coding as well as the precoding over the reals, we outline the application of IF source coding for the C-RAN scenario assuming a real channel scenario. We then comment on the adaptation of the scheme to the more realistic scenario of a complex channel.

Consider the C-RAN scenario where M transmitters send their data (that is modeled as an i.i.d. Gaussian source vector) over a real $K \times M$ MIMO broadcast channel $\mathbf{H} \in \mathbb{R}^{K\times M}$.The data is received at K receivers (relays) that wish to compress and forward it for processing (decoding) at a central node via rate-constrained noiseless bit pipes.

As we wish to minimize the distortion at the central node subject to the rate constraints, this is a distributed lossy source coding problem. See depiction in Figure 1.

Here, the covariance matrix of the received signals at the relays is given by

$$\mathbf{K_{xx}} = \mathsf{SNR}\mathbf{H}\mathbf{H}^T + \mathbf{I}. \quad (46)$$

We note that we can "absorb" the SNR into the channel and hence we set $\mathsf{SNR} = 1$, so that

$$\mathbf{K_{xx}} = \mathbf{H}\mathbf{H}^T + \mathbf{I}. \quad (47)$$

We further assume that the entries of the channel matrix $\mathbf{H}$ are Gaussian i.i.d., i.e. $\mathbf{H}_{i,j} \sim \mathcal{N}(0,\sigma^2)$ for all i, j. We note that the SVD of this matrix is

$$\mathbf{H} = \tilde{\mathbf{U}}\tilde{\mathbf{\Sigma}}\tilde{\mathbf{V}}^T \quad (48)$$

where $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ belong to the CRE. We may therefore express the (random) covariance matrix as

$$\mathbf{K_{xx}} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^T, \quad (49)$$

where $\mathbf{U}$ is drawn from the CRE. It follows that the precoding matrix $\mathbf{P}$ is redundant (as we assumed that $\mathbf{P}$ is also drawn from the CRE). Hence, the performance analysis derived above holds also for the considered scenario where the relays are distributed. Thus, assuming the encoders are subject to an equal rate constraint (which they know), then for a given distortion level, the relation between compression rate of IF source coding and the guaranteed outage probability may be bounded using Theorem 1 above.

We note that just as pre-processed IF channel coding can accomplished for complex channels as described in [2], so can pre-processed IF source coding be extended to complex Gaussian sources. In describing an outage event in this case we assume that the precoding matrix is drawn from the circular unitary ensemble (CUE). The bounds derived above (replacing K with $2K$ in all derivations) for the relation between the compression rate of IF source coding and guaranteed outage probability hold for the C-RAN scenario over complex Gaussian channels $\mathbf{H} \in \mathbb{C}^{K\times M}$, where the CUE precoding can be viewed as been performed by nature.

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